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CONCERNING OF A PARTICULAR INVERSE PROBLEM

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A coefficient inverse problem for an elliptic equation with the coefficient in the form of a stationary random field is considered.

For some time now, considerable attention has been paid to studying coefficient inverse problems in both heat conduction and hydrogeology [1-4].

Let us consider the following problem of hydrogeology. A geological region is isolated where the water conductivity coefficient T is a stationary random field that has the constant mathematical expectation T_0 and dispersion DT and whose correlation function $K_\alpha(h)$ is a function of a priori known form that depends on the parameter α. Pumping out with a constant output Q is done from one (experimental) well of radius ρ. Observational wells that change their head H_i^j are located around the experimental well on several circles of radii r_i, where j is the number of the observational well on the circle with radius r_i.

The hydrogeological process is described by the following problem for an elliptic equation in polar coordinates:

$$\frac{\partial}{\partial r}\left(T\frac{\partial H}{\partial r}\right)+\frac{T}{r}\frac{\partial H}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(T\frac{\partial H}{\partial\theta}\right)=0\,; \tag{1}$$

$$T(r,\theta)=T(r,\theta+2\pi)\,;\quad H(r,\theta)=H(r,\theta+2\pi)\,;$$

$$-\int_0^{2\pi} d\theta rT\frac{\partial H}{\partial r}\bigg|_{r=\rho}=Q\,;\quad H|_{r=R}=H_a\,, \tag{2}$$

$R \gg \rho > 0$; Q, H_a are constant; see [5].

The direct problem consists in solving Eqs. (1) and (2) with T in the form of one of the realizations of the random field.

The inverse problem consists in finding the values of T_0, DT, and α from H_i^j.

For a model example, the results of solving the direct problem were used in which the entire region was divided into blocks of size l_x (elements of inhomogeneity) on which the values of T were prescribed as independent Gaussian random values with identical T_0 and DT. These calculations were performed by a grid method according to [6].

An algorithm for solving the coefficient inverse problem is presented as follows. Let $T = T_0 + \check{T}$, $H = H_0 + \check{H}$, $Q = Q_0 + \check{Q}$, where $\check{T}$, $\check{H}$, and $\check{Q}$ are random components. According to [5], from Eqs. (1) and (2) we obtain that T_0, H_0, Q_0, satisfy Eqs. (1) and (2). The assumption that T_0 = const yields $H_0(r,\theta) = H_0(r)$, and we have

$$\check{Q}=-r\left[\frac{\partial H_0}{\partial r}\int_0^{2\pi} d\theta\check{T}+T_0\frac{\partial}{\partial r}\int_0^{2\pi} d\theta\check{H}\right].$$

Let

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TABLE 1. Input Data

r_i	H_i^j
10	2.34
10	2.36
20	2.07
20	2.03
30	1.94
30	1.93
42.5	1.45
42.5	1.86
57.5	1.1
57.5	1.92

$$\check{H}_*(r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \check{H}(r,\theta)\,; \quad \check{T}_*(r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \check{T}(r,\theta)\,.$$

Then we obtain

$$H_0(r) = \frac{Q}{2\pi r T_0}\,;$$

$$\frac{d}{dr}\check{H}_* + T_0^{-1}\frac{dH_0}{dr}\check{T}_* + \frac{1}{r}T_0^{-1}\check{Q} = 0\,; \quad \check{H}_*(R) = 0\,;$$

$$\check{H}_* = -\left[\int_R^r dr'\check{T}_*(r')\frac{dH_0}{dr}T_0^{-1} + \check{Q}\int_R^r\frac{dr'}{r'}T_0^{-1}\right].$$

Since $Q = \text{const} = Q_0$, then $\check{Q} = 0$, and this yields

$$\check{H}_*(r) = \int_R^r dr'\check{T}_*(r')\frac{Q}{2\pi T_0 r'} = \frac{Q}{4\pi^2 T_0^2}\int_R^r dr'\frac{1}{r'}\int_0^{2\pi} d\theta'\check{T}(r',\theta')\,;$$

$$\check{H}_*(r_1) - \check{H}_*(r_2) = \frac{Q}{4\pi^2 T_0}\int_{r_2}^{r_1} dr'\frac{1}{r'}\int_0^{2\pi} d\theta'\check{T}(r',\theta')\,;$$

$$D\left[\frac{\check{H}_*(r_1) - \check{H}_*(r_2)}{\ln\frac{r_1}{r_2}}\right] = \frac{Q^2}{16\pi^4 T_0^4 \ln^2\frac{r_1}{r_2}}\int_{r_2}^{r_1}\frac{dr'}{r'}\int_{r_2}^{r_1}\frac{dr''}{r''}\int_0^{2\pi} d\theta'\int_0^{2\pi} d\theta'' \times$$

$$\times E\,[\check{T}(r',\theta')\,\check{T}(r'',\theta'')] = \frac{Q^2 DT}{16\pi^4 T_0^4 \ln^2\frac{r_1}{r_2}}\lambda_\alpha(r_1,r_2)\,;$$

$$\lambda_\alpha(r_1,r_2) = \int_{r_2}^{r_1}\frac{dr'}{r'}\int_{r_2}^{r_1}\frac{dr''}{r''}\int_0^{2\pi} d\theta'\int_0^{2\pi} d\theta'' K_\alpha(r',r'',\theta',\theta'')\,,$$

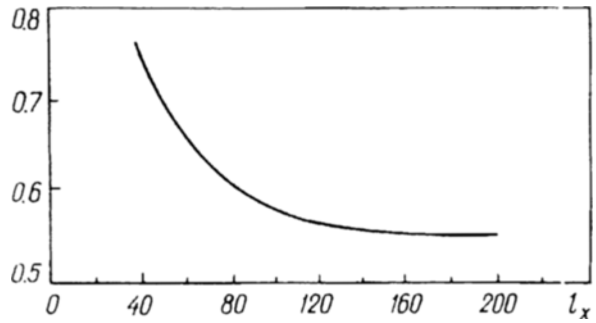

Fig. 1. Output data of the dependence of $D(DT^{l_x})/DT_0^{l_x}$ (over a vertical line) on l_x.

where $K_\alpha(r', r'', \theta', \theta'')$ is a correlation function (depending on the parameter α) of the form $K_\alpha(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X} = (r', \theta')$, $\mathbf{Y} = (r'', \theta'')$.

Now we determine T_{calc} (the calculated value of T) from the observation data for H_i^j:

$$T^{j_1, j_2}_{\text{calc}_{i_1, i_2}} = \left[\frac{2\pi}{Q}\,\frac{H_{i_1}^{j_1} - H_{i_2}^{j_2}}{\ln(r_{i_1}/r_{i_2})}\right]^{-1}.$$

Using the known values of $T^{j_1 j_2}_{\text{calc}_{i_1, i_2}}$, we calculate their mean value $T_{0\text{calc}_{i_1, i_2}}$ and dispersion $D[1/T_{\text{calc}_{i_1, i_2}}]$ for each pair i_1, i_2.

We assume that

$$D\left[\frac{\check{H}_*(r_{i_1}) - \check{H}_*(r_{i_2})}{\ln\dfrac{r_{i_1}}{r_{i_2}}}\right] = D\left[\frac{1}{T_{\text{calc}_{i_1, i_2}}}\right]\frac{Q^2}{4\pi^2}. \tag{3}$$

From Eq. (3) we obtain

$$D\left[\frac{1}{T_{\text{calc}_{i_1, i_2}}}\right] = \frac{DT\,\lambda_\alpha(r_{i_1}, r_{i_2})}{4\pi^2 T_0^4 \ln^2\dfrac{r_{i_1}}{r_{i_2}}}. \tag{4}$$

We average $T_{0\text{calc}_{i_1, i_2}}$ over all i_1, i_2 and find $T_{0\text{calc}}$ (the calculated value of the water conductivity coefficient), which will be taken as the sought value of T_0. Then we express DT from Eq. (4) and assume it to be $DT^\alpha_{i_1, i_2}$:

$$DT^\alpha_{i_1, i_2} = D\left[\frac{1}{T_{\text{calc}_{i_1, i_2}}}\right]\frac{4\pi^2 T_0^4 \ln^2(r_{i_1}/r_{i_2})}{\lambda_\alpha(r_{i_1}, r_{i_2})}.$$

Having averaged $DT^\alpha_{i_1, i_2}$ over all i_1, i_2, we obtain the mean value $(DT)_0^\alpha$ and the dispersion $D(DT^\alpha)$.

In model calculations l_x played the role of α and the ratio $D(DT^{l_x})/DT_0^{l_x}$ was plotted as a function of l_x.

We present (for control) the input and output data of the calculation of one model variant.

Having 10 observational wells with r_i and H_i^j from Table 1 as the input data, we obtain (as the output data) $T_{0\text{calc}} = 730$ (see Fig. 1). In Fig. 1 the curve to the right of the sought value of l_x approaches a constant value (within the accuracy of the graph).

In the example given, $l_x = 120$ and $DT = 8.02 \cdot 10^7$.

Comparison with the initial data of the direct problem showed the accuracy to be within the limits of the accuracy of the hydrogeological measurements and the numerical approximations, with the computer program operating an insignificantly small time compared to the solution of the direct problem by the grid method.

Since the proposed algorithm for the inverse problem employs only dispersions from the input data for H_i^j, then it is evident that small fluctuations in H_i^j will lead to small fluctuations in the output values, so that the proposed formulation of the inverse problem is correct.

NOTATION

T, water conductivity coefficient; T_0 and DT, mean value and dispersion of the water conductivity coefficient; $K_\alpha(h)$, correlation function; Q, output; r_i, radii; H_i^j, measurement of the head in the j-th well located on the circle of radius r_i; θ, angle in polar coordinates; ρ, well radius; R, radius of the contour of thje maintained supply; l_x, linear dimensions of the inhomogeneity element; $\check{T}$, $\check{H}$, $\check{Q}$, random components of the coefficient of water conductivity, the head, and the output.

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